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Localized plastic buckling deformation in axially compressed cylindrical shells

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SUMMARY

A linear partial differential equation is derived to describe growth of an axisymmetric perturbation in the plastic buckling of axially compressed cylindrical shells. Simple J_2 flow theory is used along with rigid-plastic material behavior. An asymptotic solution is then constructed for large values of a parameter which characterizes localization of an initial imperfection. The perturbation remains localized. The solution is compared with experimental results.

1. Introduction

Plastic buckling of axially compressed cylindrical shells has received considerable attention. A fairly comprehensive survey of the literature on this problem, up to 1972, is included in a general survey of plastic buckling compiled by Sewell [1]. A more recent paper by Bruhns [2] contains additional references. Much of the analysis has been based on bifurcation criteria for determining critical buckling loads, with sinusoidal buckling modes being assumed.

The method followed in the present investigation has its origin in the work of Goodier et. al. [3, 4] on dynamic plastic buckling which treats growth of a perturbation. In perturbation analysis based on the von Mises yield condition, the change in direction of the plastic strain-rate vector as the perturbation grows is the dominant effect, not work-hardening. Some experimental confirmation of this result has been noted[3, 5]. Another aspect of plastic buckling of shells which can be treated readily by perturbation analysis, and which is the subject of the present investigation, is the localization of the buckling deformation, a feature of plastic buckling long recognized experimentally. For example, a figure in the well-known reference by Timoshenko and Gere [6] illustrates that buckles in an axially compressed cylindrical shell form and collapse consecutively and not simultaneously as implied by the common assumption of a sinusoidal buckling mode.

In experiments on plastic buckling of axially compressed shells, constraint at the boundary usually governs initiation of buckling. A numerical analysis, based on a variational principle, of buckling of an axially compressed cylindrical shell with edge constraint has been given by Murphy and Lee [7]. More recently, the present author [8] has made an analytic study of localized plastic edge buckling in axially compressed steep truncated conical shells. A sufficiently large initial imperfection in an axially compressed cylindrical shell can initiate plastic buckling away from the ends. The buckling deformation remains localized to the immediate neighborhood of the imperfection.

The present paper gives an analysis of this problem that yields a prediction on the extent of the buckling deformation which is in reasonably close agreement with the results of some simple experiments. A moderately thick shell is considered, such that the critical stress for bifurcation in an initially perfect elastic-plastic cylindrical shell would be far above the yield stress. The uniform stress at the maximum load observed in tests is then a small fraction of the bifurcation stress, and linearized regular perturbation analysis based on rigid-plastic material behavior is appropriate, at least in the early stages of buckling when the deformation mode becomes established. Also in moderately thick cylindrical shells considered here, the buckling mode is axisymmetric, consisting of a single outward bulge, Fig. 1.

2. Derivation of a linearized differential equation to describe axisymmetric plastic buckling in an axially compressed infinite cylindrical shell

Except for minor changes, the notation of Timoshenko and Gere [9] is followed. The coordinates are x, θ, z: the x axis lies along a meridian, while the z axis points in the radial direction, positive inward. The nominal radius of the middle surface is a, the shell thickness is denoted by h, and the radial displacement is w, positive in the direction of positive z. M_x and M_θ are the bending moments, and N_x and N_θ are the membrane forces associated with the buckling deformation, that is, the total membrane force on the cut section x = constant is $(-Ph + N_x)$ per unit length, where P is the uniform axial compressive stress.

Constitutive relations for M_x, M_θ, N_x, and N_θ in terms of w are now derived, based on the usual assumptions of simple shell theory, and on rigid-plastic material behavior. For axisymmetric deformation, a shell lamina is in a state of plane stress with σ_x, σ_θ as principal stresses. The constitutive equation for the strain rates $\dot{\epsilon}_x$, $\dot{\epsilon}_\theta$ based on the von Mises yield condition and normality condition during 'loading' is [10],

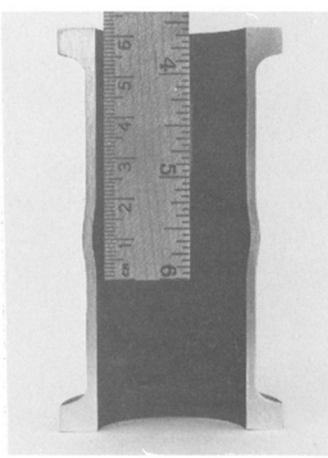

Fig. 1. Cross-section of a buckled specimen.

$$\begin{bmatrix} \dot{\epsilon}_x \\ \\ \dot{\epsilon}_\theta \end{bmatrix} = \Lambda \begin{bmatrix} \partial J_2/\partial\sigma_x \\ \\ \partial J_2/\partial\sigma_\theta \end{bmatrix} \tag{2.1}$$

where

$$\Lambda = \lambda\left(\frac{\partial J_2}{\partial\sigma_x}\,\dot{\sigma}_x + \frac{\partial J_2}{\partial\sigma_\theta}\,\dot{\sigma}_\theta\right) > 0,$$

$$\lambda = 3/(4HJ_2),$$

$$J_2 = (\sigma_x^2 - \sigma_x\sigma_\theta + \sigma_\theta^2)/3,$$

and H is the hardening modulus. Superposed dots denote differentiation with respect to time t. At least in the early stages, plastic buckling proceeds under increasing axial load and the perturbation remains small. Only the condition that $\dot{J}_2 > 0$ is considered. Equation (2.1) is linearized by putting

$$\sigma_x = -P + s_x, \qquad \sigma_\theta = s_\theta, \tag{2.2}$$

$$\epsilon_x = -\epsilon + e_x + z\kappa_x, \qquad \epsilon_\theta = \tfrac{1}{2}\epsilon + e_\theta + z\kappa_\theta. \tag{2.3}$$

In (2.3), ϵ is the uniform compressive strain due to the uniform uniaxial compressive stress P; e_x, e_θ are middle-surface strain perturbations associated with s_x, s_θ; and κ_x, κ_θ are middle-surface curvature changes. Since (2.1) holds in the absence of the perturbation,

$$H\dot{\epsilon} = \dot{P}.$$

Retention of just first order terms in s_x, s_θ yields the relations, after integration over the shell thickness,

$$Hh\dot{e}_x = \dot{N}_x - \tfrac{1}{2}\dot{N}_\theta, \tag{2.4}$$

$$Hh\dot{e}_\theta = \frac{3}{4}\,\frac{\dot{P}}{P}\,N_\theta - \tfrac{1}{2}\dot{N}_x + \tfrac{1}{4}\dot{N}_\theta. \tag{2.5}$$

Retention of first order terms in s_x, s_θ, multiplication by z, and integration over the shell thickness yield

$$\frac{Hh^3}{12}\,\dot{\kappa}_x = \dot{M}_x - \tfrac{1}{2}\dot{M}_\theta, \tag{2.6}$$

$$\frac{Hh^3}{12}\,\dot{\kappa}_\theta = \frac{3}{4}\,\frac{\dot{P}}{P}\,M_\theta - \tfrac{1}{2}\dot{M}_x + \tfrac{1}{4}\dot{M}_\theta. \tag{2.7}$$

As long as the external load is monotonically increasing in time, the loading rate can be specified arbitrarily. Equations (2.5) and (2.7) can be simplified by taking $P = P_0 e^t$. Then $\dot{P}/P =$

1. P_0 is not necessarily the initial yield stress of the material. P_0 can be the value of stress at any point along the stress-strain curve at or beyond the initial yield point. The growth of the perturbation is followed for an interval of time t starting at any arbitrary time in the loading history subsequent to the inception of plastic deformation.

In view of (2.3), the stress rates $\dot{\sigma}_x$, $\dot{\sigma}_\theta$ are referred to axes that rotate with the principal strain axes. These stress rates are unaffected by superposed rigid body rotation. Therefore (2.1) is valid for finite strain, with σ_x, σ_θ referring to current area and $\dot{e}_x$, $\dot{e}_\theta$ being logarithmic strain rates. The coordinate x measures distance along a meridian at $t = 0$. Then, with $w(x, t)$ a small radial displacement measured from the nominal radius a of the middle surface at $t = 0$, linearized expressions for the rates of strain and curvature change become, with primes indicating differentiation with respect to x,

$$\dot{e}_\theta = -\dot{w}/a, \quad \dot{\kappa}_x = -\dot{w}'', \quad \dot{\kappa}_\theta = -\dot{w}/a^2 \simeq 0. \tag{2.8}$$

The curvature change κ_θ is neglected compared to κ_x, as in the elastic analysis [9]. It follows from (2.6), (2.7) and (2.8) that

$$\dot{M}_x = -\frac{Hh^3}{36}(\ddot{w}'' + 3\dot{w}''). \tag{2.9}$$

In order to express N_θ in terms of w in (2.5), it is necessary to eliminate N_x. Since σ_x refers to current area, the condition of axial equilibrium can be written

$$(a - w)(-Ph + N_x) = -Pha$$

which, when linearized, becomes

$$N_x = -Ph(w/a). \tag{2.10}$$

Then substitution for N_x from (2.10), and for e_θ from (2.8), introduces into (2.5) two terms in w/a, one of order H and one of order P. Even though H is an order of magnitude smaller than Young's modulus for common ductile metals, the stress P would still be at least an order of magnitude smaller than H, so the term in P coming from (2.10) is neglected. Equation (2.5) is rewritten as

$$\frac{-Hh}{a}\dot{w} = \tfrac{3}{4}N_\theta + \tfrac{1}{4}\dot{N}_\theta. \tag{2.11}$$

It is of some interest to note that, if w, M_x, $N_\theta \propto e^t$, and H is identified with Young's modulus E, then (2.9), (2.11) become formally the same as the corresponding equations for an elastic shell with Poisson's ratio of one-half.

Equations (2.9) and (2.11) can now be combined with the equation of radial equilibrium [9]

$$M_x'' - Phw'' + \frac{1}{a}N_\theta = 0 \tag{2.12}$$

to obtain a single partial differential equation for w. The hardening modulus H depends on time since H is a function of P. However, H is treated as constant in time by putting $H(P) = H(P_0)$, implying that the actual stress-strain curve is approximated by the tangent through the point P_0. Equation (2.9) can be integrated with respect to time to obtain M_x. An arbitrary function of x must then be included in the expression for M_x.

Dimensionless quantities β, ψ, ξ are introduced now by putting

$$\beta^2 = 6a/h, \quad \psi = 3P_0a/(2Hh), \quad \xi = \beta x/a. \tag{2.13}$$

Then, with $W(\xi, t) = w(x, t)$, and taking account of the time dependence of P in (2.12), the following equation is obtained from (2.9), (2.11) and (2.12):

$$(\dddot{W} + 6\ddot{W} + 9\dot{W})'''' + 4\psi e^t(\ddot{W} + 4\dot{W})'' + 4\dot{W} = Q(\xi) \tag{2.14}$$

where primes now denote differentiation with respect to ξ, and $Q(\xi)$ depends on the values of M_x and w at $t = 0$. $Q(\xi)$ can be evaluated directly in terms of W by putting $t = 0$ on the left side of (2.14). Equation (2.14) is inhomogeneous if either $W(\xi, 0)$, $\dot{W}(\xi, 0)$ or $\ddot{W}(\xi, 0) \neq 0$, and these three functions of ξ can be specified arbitrarily as initial conditions on (2.14).

4. General solution for W

A general solution of (2.14) is now constructed by means of the infinite series

$$W(\xi,t) = \sum_{n=0}^{\infty} e^{nt}\, W_n(\xi,t) \tag{3.1}$$

and convergence of the series established. Substitution of this series in (2.14), and the equating of coefficients of e^{nt}, yield an infinite system of partial differential equations with constant coefficients. For $n = 0$,

$$[\dddot{W}_0 + 6\ddot{W}_0 + 9\dot{W}_0]'''' + 4\dot{W}_0 = Q(\xi), \tag{3.2}$$

and for $n \geqslant 1$,

$$\begin{aligned}&[\dddot{W}_n + 2(n+3)\ddot{W}_n + (n+3)^2 \dot{W}_n]'''' + 4[\dot{W}_n + nW_n] \\ &= -4\psi[\ddot{W}_{n-1} + (n+3)\dot{W}_{n-1}]''.\end{aligned} \tag{3.3}$$

An initial imperfection or disturbance that is even in ξ will be considered. W and W_n will then be even in ξ, and hence expressible as Fourier cosine integrals. With $A(\alpha, t)$ denoting the Fourier cosine transform [11] of $W(\xi, t)$, $A_n(\alpha, t)$ the transform of $W_n(\xi, t)$, and $R(\alpha)$ the transform of $Q(\xi)$, the series (3.1) becomes

$$A(\alpha,t) = \sum_{n=0}^{\infty} e^{nt} A_n(\alpha,t) \tag{3.4}$$

and (3.2), (3.3) become respectively

$$\ddot{A}_0 + (6 + 4/\alpha^4)\dot{A}_0 + 9A_0 = R/\alpha^4, \quad n = 0, \tag{3.5}$$

$$\ddot{A}_n + [2(n+3) + 4/\alpha^4]\dot{A}_n + [(n+3)^2 + 4n/\alpha^4]A_n$$

$$= (4\psi/\alpha^2)[\dot{A}_{n-1} + (n+3)A_{n-1}], \quad n \geqslant 1. \tag{3.6}$$

Homogeneous solutions for A_n have the form

$$A_n = G_n e^{-(n-r)t} + K_n e^{-(n-s)t}, \quad n \geqslant 0$$

where G_n, K_n are suitable arbitrary functions of α, and

$$\left.\begin{matrix} r \\ s \end{matrix}\right\} = \frac{-1}{\alpha^4}\,[1 \mp (1 + 3\alpha^4)^{\frac{1}{2}}]^2 = -(3 + 2/\alpha^4) \pm (2/\alpha^4)(1 + 3\alpha^4)^{\frac{1}{2}}. \tag{3.7}$$

For $\alpha \ll 1$,

$$r = \frac{-9}{4}\,\alpha^4 + \frac{27}{8}\,\alpha^8 + \ldots, \qquad s = \frac{-4}{\alpha^4} - 6 + \frac{9}{4}\,\alpha^4 + \ldots \tag{3.8}$$

Hence r has a zero of order 4 at the origin $\alpha = 0$, and s has a pole of order 4; $r \leqslant 0$ while $s < 0$, for all real α. Both r, s have monotonic derivatives, and $r, s \to -3$ as $\alpha \to \infty$.

The complete solution for $A(\alpha, t)$ must contain three arbitrary functions to satisfy the initial conditions on $W(\xi, t)$. It is convenient to assign these functions to $A_0(\alpha, t)$ and put

$$A_0(\alpha,t) = G_0(\alpha)e^{rt} + K_0(\alpha)e^{st} + R_0(\alpha) \tag{3.9}$$

where G_0, K_0, R_0 are arbitrary. The solutions of (3.6) for A_n are written as recursion relations in the form of convolution integrals,

$$A_n(\alpha,t) = \frac{4\psi}{\alpha^2} \int_0^t A_{n-1}(\alpha,\tau) g_n(\alpha, t-\tau)\,d\tau, \quad n \geqslant 1 \tag{3.10}$$

where

$$g_n(\alpha,t) = \frac{r+3}{r-s}\,e^{-(n-r)t} - \frac{s+3}{r-s}\,e^{-(n-s)t}. \tag{3.11}$$

It is easily shown, using (3.7), that

$$0 \leqslant g_n(\alpha,t) \leqslant g_n(\alpha,0) = 1, \quad 0 \leqslant \alpha < \infty, \quad 0 \leqslant t < \infty. \tag{3.12}$$

From (3.10), it follows that

$$A_n(\alpha,0) = 0, \quad n \geqslant 1 \quad \text{and} \quad \dot{A}_n(\alpha,0) = 0, \quad n \geqslant 2 \tag{3.13}$$

while the initial value of $\dot{A}_1$ is given by

$$\dot{A}_1(\alpha,0) = (4\psi/\alpha^2)\, A_0(\alpha,0). \tag{3.14}$$

The functions G_0, K_0, R_0 can now be found in terms of $A(\alpha, 0)$, $\dot{A}(\alpha, 0)$, $\ddot{A}(\alpha, 0)$. From the Fourier cosine transform of (2.14) and (3.5), R_0 is determined as

$$9R_0 = \ddot{A}(\alpha,0) + (6 + 4/\alpha^4 - 4\psi/\alpha^2)\, \dot{A}(\alpha,0) + (9 - 16\psi/\alpha^2)\, A(\alpha,0). \tag{3.15}$$

Also, from (3.4), noting (3.13) and (3.14),

$$A(\alpha,0) = A_0(\alpha,0), \tag{3.16}$$

$$\dot{A}(\alpha,0) = \dot{A}_0(\alpha,0) + (4\psi/\alpha^2)A_0(\alpha,0). \tag{3.17}$$

When A_0, $\dot{A}_0$ are expressed in terms of G_0, K_0, R_0, (3.16), (3.17) become two equations for determining G_0, K_0. Then, substitution for G_0, K_0 back in (3.9) yields the result

$$A_0(\alpha,t) = A(\alpha,0)\,[T_1 - (4\psi/\alpha^2)T_2] + \dot{A}(\alpha,0)T_2 + R_0[1 - T_1] \tag{3.18}$$

where

$$T_1 = (re^{st} - se^{rt})/(r - s),$$

$$T_2 = (e^{rt} - e^{st})/(r - s).$$

For real α, e^{st} goes strongly to zero as $\alpha \to 0$, and T_2, $(1 - T_1)$ behave approximately as

$$T_2 = \alpha^4/4, \quad 1 - T_1 = (9/4)\alpha^4 t, \quad \alpha \to 0.$$

Thus T_2 has a zero of order 4 which cancels the pole of order 2 appearing in its coefficient in the first term on the right side of (3.18); $(1 - T_1)$ also has a zero of order 4 which cancels the poles in R_0 which appear on the right side of (3.15). Hence $A_0(\alpha, t)$ is bounded for real α provided $A(\alpha, 0)$, $\dot{A}(\alpha, 0)$, $\ddot{A}(\alpha, 0)$ are.

The convergence of the series representation for $W(\xi, t)$, (3.1), is now demonstrated by showing that the sum

$$\sum_{n=1}^{\infty} e^{nt} \int_0^{\infty} |A_n(\alpha,t)|\, d\alpha$$

converges. From (3.10), (312) it follows that

$$|A_n(\alpha,t)| \leqslant \frac{4\psi}{\alpha^2} |A_{n-1}|_{\max} \int_0^t g_n(\alpha,t.-\tau)d\tau \tag{3.19}$$

where $|A_{n-1}|_{\max}$ refers to the maximum of the absolute value of $A_{n-1}(\alpha,\tau)$ for $0 \leqslant \alpha < \infty$, $0 \leqslant \tau \leqslant t$. Then, writing I_n, $n \geqslant 1$, for the integral in (3.19),

$$I_n = \frac{r+3}{(r-s)(n-r)}[1-e^{-(n-r)t}] - \frac{s+3}{(r-s)(n-s)}[1-e^{-(n-s)t}] \tag{3.20}$$

$$\leqslant \frac{r+3}{(r-s)(n-r)} - \frac{s+3}{(r-s)(n-s)} = \frac{(n+3)\alpha^4}{4n+(n+3)\alpha^4}\ .$$

Then (3.19) can be rewritten

$$|A_n(\alpha,t)| \leqslant \frac{4\psi(n+3)\alpha^2}{4n+(n+3)^2\alpha^4} |A_{n-1}|_{\max}. \tag{3.21}$$

The coefficient of $|A_{n-1}|_{\max}$ has a single maximum when $\alpha^2 = 2n^{\frac{1}{2}}/(n+3)$, and this maximum value is $\psi/n^{\frac{1}{2}}$. Hence the inequalities can be formed:

$$|A_n(\alpha,t)| \leqslant \frac{\psi}{n^{\frac{1}{2}}} |A_{n-1}|_{\max} \leqslant \frac{\psi}{(n!)^{\frac{1}{2}}} |A_0|_{\max}. \tag{3.22}$$

Then, from (3.21), using (3.22),

$$|A_n(\alpha,t)| \leqslant \frac{4\psi(n+3)\alpha^2}{4n+(n+3)^2\alpha^4} \frac{\psi^{n-1}}{[(n-1)!]^{\frac{1}{2}}} |A_0|_{\max}. \tag{3.23}$$

The right side of (3.23) has simple poles where $\alpha = (\pm 1 \pm i)n^{\frac{1}{4}}/(n+3)^{\frac{1}{2}}$. The residue theorem yields

$$\int_0^\infty |A_n(\alpha,t)|\, d\alpha \leqslant \frac{\pi\psi^n}{n^{\frac{1}{4}}(n+3)^{\frac{1}{2}}[(n-1)!]^{\frac{1}{2}}} |A_0|_{\max}$$

$$< \frac{\pi\psi^n}{(n!)^{\frac{1}{2}}} |A_0|_{\max}, \quad n \geqslant 1 \tag{3.24}$$

and hence

$$\sum_{n=1}^{\infty} e^{nt} \int_0^\infty |A_n(\alpha,t)|\, d\alpha < \pi |A_0|_{\max} \sum_{n=1}^{\infty} \frac{(\psi e^t)^n}{(n!)^{\frac{1}{2}}} \tag{3.25}$$

the infinite series on the right converging for all ψ and t. W is then obtained as the inverse transform of (3.4).

4. Asymptotic solution for large time for a local axisymmetric initial imperfection

As a preliminary step to obtaining a solution for arbitrary time, it is useful to establish the asymptotic form of the solution of (2.14) as $t \to \infty$ for a particular set of initial conditions. As an initial imperfection or disturbance, it is convenient to choose $W(\xi, 0)$, $\dot{W}(\xi, 0)$, or $\ddot{W}(\xi, 0)$ proportional to $\exp(-b\xi^2)$. For an imperfection or disturbance localized to the neighborhood of the origin, b is large. Three sets of initial conditions can be applied to (2.14);

Case 1. $W(\xi,0) = e^{-b\xi^2}, \quad \dot{W}(\xi,0) = \ddot{W}(\xi,0) = 0;$

Case 2. $\dot{W}(\xi,0) = e^{-b\xi^2}, \quad W(\xi,0) = \ddot{W}(\xi,0) = 0;$

Case 3. $\ddot{W}(\xi,0) = e^{-b\xi^2}, \quad W(\xi,0) = \dot{W}(\xi,0) = 0.$

Since the three cases can be treated by the same method, and lead to similar results, only Case 1 describing an initial geometric imperfection will be considered in detail. For this case, (3.18) reduces to

$$A_0(\alpha,t) = A(\alpha,0)\,[(1 - 16\psi/9\alpha^2) - (4\psi/\alpha^2)T_2 + (16\psi/9\alpha^2)T_1] \tag{4.1}$$

where

$$A(\alpha,0) = (2b)^{-\frac{1}{2}} e^{-\alpha^2/4b}.$$

It is convenient first to obtain $W_0''(\xi, t)$, which is the inverse transform of $-\alpha^2 A_0(\alpha, t)$, and then proceed to determine $W_0(\xi, t)$. Application of the inverse Fourier cosine transformation to (4.1) yields

$$W_0''(\xi,t) = W''(\xi,0) + (16\psi/9)\, W(\xi,0) - (16\psi/9)I_1 + 4\psi I_2 \tag{4.2}$$

where

$$I_1 = (\pi b)^{-\frac{1}{2}} \int_0^\infty e^{-\alpha^2/4b}\, T_1(\alpha,t) \cos \xi\alpha d\alpha,$$

$$I_2 = (\pi b)^{-\frac{1}{2}} \int_0^\infty e^{-\alpha^2/4b}\, T_2(\alpha,t) \cos \xi\alpha d\alpha.$$

Since $(r - s) > 0$ for $0 \leqslant \alpha < \infty$, T_1 and T_2 are written, for $t \to \infty$,

$$T_1 = \frac{e^{rt}}{r-s}\,[re^{-(r-s)t} - s] \sim \frac{-se^{rt}}{r-s},$$

$$T_2 = \frac{e^{rt}}{r-s}\,[1 - e^{-(r-s)t}] \sim \frac{e^{rt}}{r-s}.$$

The exponential e^{rt} has a supremum at $\alpha = 0$. For $t \to \infty$, the values of the integrands near the origin then dominate I_1 and I_2, so $e^{-\alpha^2/4b}$ and $\cos \xi\alpha$ are replaced by 1, and r, s are replaced by the first terms of their power series expansions about $\alpha = 0$. Hence

$$(\pi b)^{\frac{1}{2}} I_1 \sim \int_0^\infty e^{(-9t/4)\alpha^4} d\alpha = \frac{1}{4}\left(\frac{4}{9}\right)^{\frac{1}{4}} \frac{\Gamma(1/4)}{t^{\frac{1}{4}}},$$

$$(\pi b)^{\frac{1}{2}} I_2 \sim \frac{1}{4} \int_0^\infty \alpha^4 e^{(-9t/4)\alpha^4} d\alpha = \frac{1}{16}\left(\frac{4}{9}\right)^{\frac{5}{4}} \frac{\Gamma(5/4)}{t^{\frac{5}{4}}}.$$

Therefore $I_1, I_2 \to 0$ as $t \to \infty$. Writing $W_0(\xi, \infty)$ for the asymptotic value of $W_0(\xi, t)$ as $t \to \infty$, (4.2) becomes

$$W_0''(\xi,\infty) = W''(\xi,0) + (16\psi/9)\, W(\xi,0).$$

Integrating twice with respect to ξ yields

$$W_0(\xi,\infty) = W(\xi,0) + (16\psi/9) \int_0^\xi (\xi - \eta)\, W(\eta,0) d\eta + C_1 \xi + C_0$$

where C_1, C_2 are arbitrary constant. The integral appearing on the right side can be evaluated, as $b \to \infty$, by Laplace's method [12], yielding

$$\int_0^\xi (\xi - \eta)\, W(\eta,0) d\eta = \int_0^\xi (\xi - \eta) e^{-b\eta^2} d\eta = \xi(\pi/4b)^{\frac{1}{2}}.$$

The condition that $W \equiv 0$ as $\xi \to \infty$ requires that $C_1 = -(8\psi/9)(\pi/b)^{\frac{1}{2}}$, $C_0 = 0$. Hence

$$W_0(\xi,\infty) = W(\xi,0).$$

Then, since $W_0(\xi,0) = W(\xi,0)$, from (3.16), it follows that

$$W_0(\xi,0) = W_0(\xi,\infty) = W(\xi,0)$$

so the time dependence of $W_0(\xi, t)$ appears to be negligible. It is concluded then that $A_0(\alpha, t) = A_0(\alpha, 0)$, at least for an initial imperfection localized to the neighborhood of the origin.

The recursion relation (3.10) yields the asymptotic relation, as $t \to \infty$,

$$A_n \sim \frac{4\psi(n+3)\alpha^2}{4n + (n+3)^2\alpha^4} A_{n-1}$$

and A_n can be expressed in terms of A_0 by

$$A_n = f_n(\alpha) A_0$$

where

$$f_n(\alpha) = \psi^n \prod_{m=1}^{n} \frac{4(m+3)\alpha^2}{4m + (m+3)^2\alpha^4} \quad . \tag{4.3}$$

The locations of the poles of $f_n(\alpha)$, which occur where

$$\alpha = (\pm 1 \pm i) m^{\frac{1}{4}}/(m+3)^{\frac{1}{2}}, \quad m = 1, 2, 3 \, \ldots, n$$

are independent of the axial load and hardening modulus, and depend only on the geometric quantities a and h.

The inverse transform of $f_n(\alpha)$ is denoted by $F_n(\xi)$. Application of the residue theorem to the inversion integral yields, for $\xi \geqslant 0$,

$$F_n(\xi) = -\pi^{\frac{1}{2}}(2\psi)^n \sum_{p=1}^{n} \frac{(2n-3)/4}{(p+3)^{\frac{1}{2}}} P_{np} e^{-k_p \xi} \sin\left[k_p \xi + \frac{(2n-3)\pi}{4} \right] \tag{4.4}$$

where

$$P_{np} = \prod_{\substack{m=1 \\ m \neq p}}^{n} \frac{(m+3)\,(p+3)}{m(p+3)^2 - p(m+3)^2}\,, \quad k_p = p^{\frac{1}{4}}/(p+3)^{\frac{1}{2}}.$$

The convolution theorem for the Fourier cosine transform then gives $W_n(\xi, \infty)$ as the integral

$$W_n(\xi,\infty) = \tfrac{1}{2} \int_0^\infty e^{-b\eta^2} [F_n(|\,\xi - \eta\,|) + F_n(\xi + \eta)] d\eta. \tag{4.5}$$

As $b \to \infty$, Laplace's method yields

$$W_n(\xi,\infty) \sim (\pi/4b)^{\frac{1}{2}} F_n(\xi) \tag{4.6}$$

where $F_n(\xi)$ is given by (4.4).

5. General solution for a local axisymmetric initial imperfection

An asympototic solution for $b \to \infty$, valid for arbitrary time and holding uniformly in the limits $t \to 0$ and $t \to \infty$ is now constructed. In view of (3.7), the functions $T_1(\alpha, t)$, $T_2(\alpha, t)$ which appear in the general expression for $A_0(\alpha, t)$, (3.18), can be written

$$T_1 = e^{-(3+\beta)t}[\cosh \gamma t + (3+\beta)(\sinh \gamma t)/\gamma],$$

$$T_2 = e^{-(3+\beta)t}(\sinh \gamma t)/\gamma,$$

where

$$\beta = 2/\alpha^4, \quad \gamma = 2(1 + 3\alpha^4)^{\frac{1}{2}}/\alpha^4. \tag{5.1}$$

Since T_1 and T_2 are even functions in γ, they are single-valued and analytic over the entire α plane, except at the origin $\alpha = 0$ where there is an essential singularity characterized by e^{-4t/α^4}. T_1 and T_2 are expressible as Laurent's series in powers of $1/\alpha^4$, convergent for $|\alpha| > 0$, which begin

$$T_1 = e^{-3t}[(1+3t) + 6t^3 \frac{1}{\alpha^4} + (\ldots)\frac{1}{\alpha^8} + \ldots],$$

$$T_2 = e^{-3t}[t - 2t^2(1-t)\frac{1}{\alpha^4} + (\ldots)\frac{1}{\alpha^8} + \ldots].$$

Then $A_0(\alpha,t)$, as given by (4.1), can be written

$$A_0(\alpha,t) = (2b)^{-\frac{1}{2}} e^{-\alpha^2/4b} L_0(\alpha,t) \tag{5.2}$$

where L_0 represents the Laurent's series

$$L_0(\alpha,t) = 1 - \frac{16\psi}{9}(1 - e^{-3t} - \tfrac{3}{4}te^{-3t})\frac{1}{\alpha^2} + 8\psi t^2(1+\tfrac{1}{3}t)e^{-3t}\frac{1}{\alpha^6} + \ldots \tag{5.3}$$

which converges for $|\alpha| > 0$. With the substitution $\alpha = b\zeta$, the inversion integral for $W_0(\xi,t)$ becomes

$$W_0(\xi,t) = (2\pi)^{-\frac{1}{2}} b \int_{-\infty}^{\infty} e^{i\xi b\zeta} A_0(b\zeta,t)d\zeta. \tag{5.4}$$

Since the Laurent's series L_0 appearing in A_0 does not converge when $\zeta = 0$, the path of integration along the real axis in (5.4) is indented at the origin by means of a semi-circle C of radius δ in the upper half-plane. As the semi-circle C is traversed in the clockwise direction, $\zeta = -\delta e^{-i\phi}$, with ϕ increasing from zero to π. Hence, along C,

$$|e^{b(-\zeta^2/4+i\xi\zeta)}| \leqslant e^{b\delta^2/4} e^{-\xi b\delta \sin\phi}, \quad 0 \leqslant \phi \leqslant \pi.$$

The integrand in (5.4) can be made to go exponentially to zero along C as $\delta \to 0$ by requiring that $b\delta \to \infty$ and $b\delta^2 \to 0$. Putting $\delta = b^{-2/3}$ and letting $b \to \infty$ satisfies both these requirements. Also, when $b\delta \to \infty$, $L_0 \to 1$. Hence the integral around the semi-circle C vanishes in the limit as $b \to \infty$, and the path of integration along the real axis can be deformed into any path lying in the upper half-plane, along which the order of summation and integration can be interchanged on the right side of (5.4) since L_0 converges everywhere along the path.

The method of steepest descents [12] is now applied in the term-by-term integration of the right side of (5.4). Substitution from (5.2) and (5.3) in (5.4) leads to a series beginning

$$W_0(\xi,t) = \frac{1}{2}\left(\frac{b}{\pi}\right)^{\frac{1}{2}} \left\{ \int e^{b(-\zeta^2/4+i\xi\zeta)}d\zeta + \phi(t)b^{-2}\int\zeta^{-2}e^{b(-\zeta^2/4+i\xi\zeta)}d\zeta + \ldots \right\} \tag{5.5}$$

where $\phi(t)$ is the coefficient of $1/\alpha^2$ on the right side of (5.3). The exponential functions in the integrands on the right side of (5.5) all have the same saddle-point, $\zeta = 2\xi i$, and paths of steepest descent which are straight lines parallel to the real axis through the saddle point. The asymptotic expression for $W_0(\xi, t)$ as $b \to \infty$ begins

$$W_0(\xi,t) \sim e^{-b\xi^2} \left\{ 1 + \frac{4\psi}{9} (1 - e^{-3t} - \tfrac{3}{4} t e^{-3t}) \frac{1}{b^2 \xi^2} + \ldots \right\}.$$

When just the leading term is retained, the result of Sect. 4 is recovered, namely that

$$W_0(\xi\ t) = W(\xi,0)$$

since the leading term is independent of time.

Just the leading term in L_0 is retained now in A_0 in determining A_1 by means of the recursion relation (3.10), the result being

$$A_1(\alpha,t) = 4\psi(2b)^{-\frac{1}{2}} \alpha^{-2} e^{-\alpha^2/4b} I_1(\alpha,t) \tag{5.6}$$

where I_1 is given by (3.20). Then, using (3.7), I_1 becomes

$$I_1 = \frac{\alpha^4}{1 + 4\alpha^4} \left\{ 1 - e^{-(4+\beta)t} [\cosh \gamma t + (\sinh \gamma t)/2\gamma] \right\},$$

where β, γ are defined by (5.1). Since I_1 is an even function in γ, it is single-valued over the entire α plane. The expression in the curly brackets in analytic everywhere except the origin, where there is an essential singularity, while the coefficient $\alpha^4/(1 + 4\alpha^4)$ has simple poles where $\alpha = (\pm 1 \pm i)/2$. The expression in the curly brackets has the Laurent's series expansion, convergent for $|\alpha| > 0$,

$$L_1(\alpha,t) = (1 - e^{-4t}) + (t - 6t^2) e^{-4t} \frac{1}{\alpha^4} + \ldots \tag{5.7}$$

and hence (5.6) can be written

$$A_1(\alpha,t) = (2b)^{-\frac{1}{2}} f_1(\alpha) e^{-\alpha^2/4b} L_1(\alpha,t) \tag{5.8}$$

where $f_1(\alpha)$ is defined by (4.3).

$W_1(\xi, t)$ can now be obtained easily from the right side of (5.8) by means of the convolution theorem. The method used in obtaining the inverse transform of $A_0(\alpha, t)$ can be applied to the expression $(2b)^{-\frac{1}{2}} e^{-\alpha^2/4b} L_1(\alpha,t)$. $f_1(\alpha)$ has not been included in L_1, since then L_1 would not converge for arbitrarily small α. However, the inverse transform of $f_1(\alpha)$ has already been obtained; it is $F_1(\xi)$, given by (4.4). Since just the leading term in L_0 was retained, only the leading term in L_1 can now be retained, and the convolution theorem gives

$$W_1(\xi,t) \sim \tfrac{1}{2}(1 - e^{-4t}) \int_0^\infty e^{-b\eta^2} [F_1(|\xi - \eta|) + F_1(\xi + \eta)] d\eta. \tag{5.9}$$

The leading term in the asymptotic expansion for $W_1(\xi, t)$ as $b \to \infty$ is then obtained from (5.9) by Laplace's method. Hence, noting (4.6),

$$W_1(\xi,t) \sim (\pi/4b)^{\frac{1}{2}} F_1(\xi)(1 - e^{-4t}) = W_1(\xi,\infty)(1 - e^{-4t}). \tag{5.10}$$

Since the leading term in L_1 is time-dependent, the procedure in obtaining A_2 from A_1 differs somewhat from the determination of A_1 from A_0. When just the leading term in L_1 is retained in (5.7), substitution from (5.7) and (5.8) in the recursion relation (3.10) and integration yield

$$A_2(\alpha,t) = (4\psi\alpha^{-2})(2b)^{-\frac{1}{2}} f_1(\alpha) e^{-\alpha^2/4b}[I_2(\alpha,t) + J_2(\alpha,t)] \tag{5.11}$$

where

$$I_2(\alpha,t) = \frac{5\alpha^4}{8 + 25\alpha^4}\left\{1 - e^{-(5+\beta)t}[\cosh\gamma t + \beta(\sinh\gamma t)/5\gamma]\right\},$$

$$J_2(\alpha,t) = \frac{-\alpha^4}{\alpha^4 - 8}\left\{e^{-4t} - e^{-(5+\beta)t}[\cosh\gamma t + 5\beta(\sinh\gamma t)/\gamma]\right\},$$

and β, γ are again defined by (5.1). The expressions in the curly brackets are expressible as Laurent's series in powers of $1/\alpha^4$ which are convergent for $|\alpha| > 0$. I_2 and J_2 can be written

$$\begin{aligned} I_2(\alpha,t) &= \frac{5\alpha^4}{8 + 25\alpha^4}\left\{1 - e^{-5t} + (\ldots)\frac{1}{\alpha^4} + \ldots\right\}, \\ J_2(\alpha,t) &= \frac{-\alpha^4}{\alpha^4 - 8}\left\{e^{-4t} - e^{-5t} + (\ldots)\frac{1}{\alpha^4} + \ldots\right\}. \end{aligned} \tag{5.12}$$

Just the leading terms in the Laurent's series in the expressions for I_2 and J_2, (5.12), contribute to the leading term in the asymptotic expansion of $W_2(\xi, t)$ for large b. Hence, for $b \to \infty$,

$$A_2(\alpha,t) \sim (2b)^{-\frac{1}{2}} e^{-\alpha^2/4b}\left[f_2(\alpha)(1 - e^{-5t}) - \frac{4\psi\alpha^2}{\alpha^4 - 8} f_1(\alpha)(e^{-4t} - e^{-5t})\right]. \tag{5.13}$$

The inverse transform of $f_2(\alpha)$ is $F_2(\xi)$, defined by (4.4). The second term in the square bracket in (5.13) has simple poles on the real axis, where $\alpha = \pm 2^{3/4}$, which arise as a result of truncating the Laurent's series in the expression for J_2, (5.12). The exact expression for J_2, (5.11), has an essential singularity at the origin as its only singularity. The inverse transform of $4\psi\alpha^2 f_1(\alpha)/(\alpha^4$

$-$ 8) is denoted by $K_2(\xi)$, and is obtained as the Cauchy principal value of the inversion integral. Hence

$$K_2(\xi) = \frac{8\sqrt{\pi}}{33}\,\psi^2\,\{-2^{\frac{1}{4}}\exp(-2^{3/4}\xi) + \exp(-\xi/2)\sin(\xi/2 + \pi/4)\}.$$

The convolution theorem for the Fourier cosine transform and Laplace's method then yield, for $\xi > 0$,

$$W_2(\xi,t) \sim (\pi/4b)^{-\frac{1}{2}}[F_2(\xi)(1 - e^{-5t}) - K_2(\xi)(e^{-4t} - e^{-5t})]. \tag{5.14}$$

Numerical calculation shows that

$$|F_2(\xi)| \leqslant |F_2(0)| \approx 0.46\psi^2,$$
$$|K_2(\xi)| \leqslant |K_2(0)| \approx 0.21\psi^2.$$

Also,

$$(e^{-4t} - e^{-5t}) \leqslant 0.2(1 - e^{-5t}).$$

The upper bound on the term in K_2 is less than 10 percent of the corresponding upper bound on the term in F_2; also, K_2 dies out more rapidly for increasing ξ than F_2, and this term dies out with increasing time. Therefore, with small error,

$$W_2(\xi,t) \sim (\pi/4b)^{-\frac{1}{2}}F_2(\xi)(1 - e^{-5t}) = W_2(\xi,\infty)(1 - e^{-5t}).$$

Determination of the remaining A_n for $n \geqslant 3$ can be carried out following the same procedure as used for A_2, and leads to the general result

$$W_n(\xi,t) \sim (\pi/4b)^{\frac{1}{2}}F_n(\xi)[1 - e^{-(n+3)t}] = W_n(\xi,\infty)[1 - e^{-(n+3)t}], \quad n \geqslant 1. \tag{5.15}$$

Apart from the common factor $(\pi/4b)^{\frac{1}{2}}$, the functions $F_n(\xi)$ in the approximate formula for $W_n(\xi, t)$ are upper bounds on $W_n(\xi, t)$ which are approached asymptotically as $t \to \infty$, while the approach is uniform over ξ. The extent of the plastic buckling deformation is thus governed by the functions $F_n(\xi)$ and, in view of (4.4), by the factors $\exp(-k_p\xi)$. The first few values of k_p are given in Table 1.

TABLE 1

p	1	2	3	4	5	6	7	8	9	10
k_p	0.50	0.53	0.54	0.53	0.53	0.52	0.51	0.51	0.50	0.49

Thus k_p rises to a maximum when $p = 3$, then slowly diminishes to zero as p increases. For $n \leqslant 9$, the terms W_n die out in ξ at least as fast as $\exp(-k_1\xi)$, so these first few terms describe deformation localized to the neighborhood of the origin. For $n \to \infty$, W_n describe deformation that spreads farther and father from the origin. However, from (3.24), it follows that the terms W_n diminish faster than $(n!)^{-\frac{1}{2}}$. For $n \geqslant 10$, $(n!)^{-\frac{1}{2}} < 10^{-3}$, so terms beyond the first few which describe a localized deformation are insignificant. Typically, ψ would be of order one or less.

6. Comparison with experiment

A number of specimens were made from a structural aluminum alloy, 6061-T6. A stress-strain curve for this material is shown in [5], and has the feature of a rather sharply defined yield point with an essentially constant tangent modulus beyond the yield point. The specimens were 100 mm long, with heavy flanges at both ends. The specimens had an original middle-surface radius $a = 20$ mm and thickness $h = 2.5$ mm. Various artificial imperfections were tried. It was found that, while the analysis tolerates the limiting process of letting $b \to \infty$, the experiments did not. The extent of the imperfection in the x direction must be of order $(ah)^{\frac{1}{2}}$. More localized imperfections on the order of the thickness h were ineffective. In cases when a buckle did not form at the location of the artificial imperfection, it formed adjacent to a flange, constraint at the boundary then being the dominant effect in initiating buckling.

An artificial imperfection which yielded reproducible results consisted of a slight gradual thinning of the shell made by cutting a shallow circumferential groove about 10 mm wide and 0.1 mm deep at its deepest point. These imperfections were so slight that they were virtually imperceptible to the unaided senses of sight and touch. Outward buckling occurred regardless of whether the groove was on the inside or outside. The thinning of the shell likely produced a local outward perturbation in the prebuckling deformation. There would be a local increase in the axial compressive strain $-\epsilon_x$ which in turn would be related to the radial displacement w by

$$-w/a = \epsilon_\theta = -\epsilon_x/2.$$

Buckles initiated by thinning had the same shape and extent as those which formed adjacent to a flange, so the slight variation in h due to the thinning is not considered to affect buckling once it is underway. In all cases, whether the buckle formed at the site of the imperfection or adjacent to a flange, it consisted of a single outward axisymmetric bulge, Fig. 1.

The functions F_n, which describe the radial displacement field, have been calculated for $n = 1$ to 4 using formula (4.4) and the numerical data: $a = 20$ mm, $h = 2.5$ mm and $\psi = 1$. A more accurate value of ψ for these specimens would be $\psi = 0.8$ but, since the result sought does not depend strongly on ψ, the value $\psi = 1$ was used for convenience. The functions F_1 to F_4 are shown plotted in Fig. 2. The horizontal scale for ξ has been carefully matched to the scale of the specimen cross-section traced from a photographic enlargement, and shown at the bottom of Fig. 2. No attempt has been made to relate the vertical scale to the actual radial displacement of the specimen. The decay of the functions F_n with increasing ξ roughly agrees with that of the deformation of the specimen. However, the observed deformation is strictly outwards and

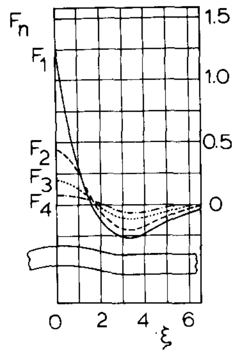

Fig. 2. Plot of functions $F_n(\xi)$ versus the dimensionless coordinate ξ. The outline of the cross-section of a buckled specimen is shown at the bottom to the same horizontal scale.

does not show the waviness exhibited by the graphs of the functions F_1, .., F_4 in Fig. 2. This discrepancy is thought to be due to neglecting in the analysis, not the elastic strains, but the effect of transverse shear on the deformation.

7. Discussion

The rigid-plastic idealization of material behavior used in conjunction with perturbation analysis here and in other cases [3, 4, 5, 8] provides a good description of plastic buckling deformation observed in experiments, especially the localized deformation resulting from edge constraint or initial imperfections. Although neglecting elastic strains precludes determining the buckling load as an eigenvalue, the present approach does permit calculation of the increase in deflection accompanying a given increase in the applied load. Since it has long been recognized that bifurcation analysis based on simple J_2 flow theory yields predictions on the buckling load that are too high, rigid-plastic perturbation analysis could be a useful alternative in practical situations of plastic buckling.

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